Fluid flow through porous media

ONE DIMENSIONAL FINITE ELEMENT FORMULATION
USING
DIRECT AND VARIATIONAL APPROACH

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General Steps to be followed while solving a problem on Fluid flow through porous media by using FEM:

1. Discretize and select the element type
2. Choose a potential function
3. Define the gradient / potential and velocity / gradient relationship
4. Derive the element stiffness matrix and equations
5. Assemble the element equations to obtain the global equations and introduce boundary conditions
6. Solve for the nodal potential
7. Solve for the element velocities and volumetric flow rates using

$$Q_f = v \, x \, A \quad m^3/s$$

Points to be remembered:

1. This is similar to one dimensional heat conduction problem
2. The temperature function T is to be replaced by fluid velocity potential Φ
3. The nodal temperature vector $\{t\}$ should be replaced by vector of nodal potential denoted by $\{p\}$
4. Fluid velocity v replaces heat flux q and permeability coefficient K for flow through porous medium replaces the conduction coefficient K
5. If fluid flow through a pipe or around a solid body is considered, then K is taken as unity

Equation for 1-D fluid flow through a porous medium is,

$$\frac{d}{dx}\left(K_{xx}\frac{d\Phi}{dx}\right)+\bar{Q}=0 ------(1)$$

Where, $\bar{Q}=\frac{Q}{A\,dx}$ is the volume flow rate per unit volume in $\frac{1}{s}$

K_{xx} = permeability coefficient

If Kxx = a constant, then

$$K_{xx}\frac{d^2\Phi}{dx^2}+\bar{Q}=0------(2)$$

The B.C.'s are, $\Phi=\Phi_B$ on S_1 Where Φ_B = boundary fluid head

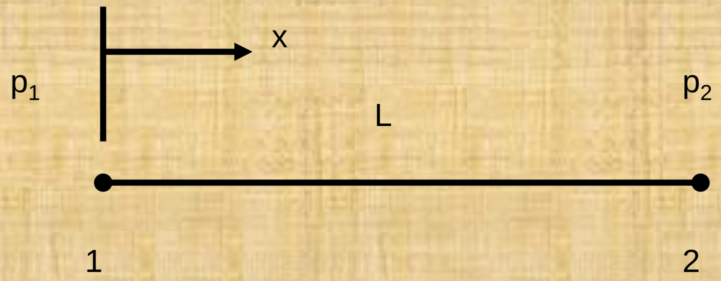
and $v_x^* = -K_{xx}\frac{d\Phi}{dx}$ = constant on S_2

v_x^* = prescribed velocity and $\frac{d\Phi}{dx}$ = gradient

If the surface is impermeable then $v_x^* = 0$

STEPS TO BE FOLLOWED:

Step 1. Select element type.



One –D element

Step 2. Choose a potential function.

$$\Phi = N_1 p_1 + N_2 p_2 ---------(1)$$

Where,

$$N_1 = 1 - \frac{x}{L}, \quad N_2 = \frac{x}{L} ---(2)$$

In matrix form,

$$[N] = \begin{bmatrix} 1 - \frac{x}{L} & \frac{x}{L} \end{bmatrix} \text{ and } \{p\} = \begin{Bmatrix} p_1 \\ p_2 \end{Bmatrix} ------(3)$$

$$\{\Phi\} = [N]\{p\} ------(4)$$

Step 3. Define the gradient / potential and velocity / gradient relationships

The hydraulic gradient matrix is,

$$\{g\} = \left\{ \frac{d\Phi}{dx} \right\} = [B]\{p\} - - - - - - - - - - - - (5)$$

Where,

$$[B] = \left[\frac{dN_1}{dx} \quad \frac{dN_2}{dx} \right] = \left[-\frac{1}{L} \quad \frac{1}{L} \right] - - - - - - - - - - - (6)$$

The velocity / gradient relationship,

$$v_x = -[D]\{g\} - - - - - - - - - - - (7)$$

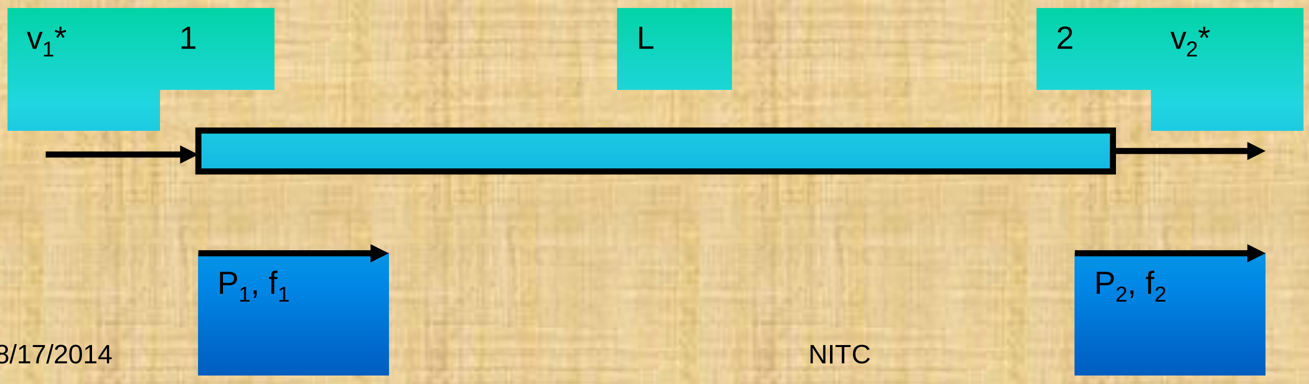
Where the material property matrix is now given by,

$$[D] = [K_{xx}] --------(8)$$

Step 4. Derive the element stiffness matrix and equations.

Consider the fluid element,

Length = L and cross sectional area = A.



Relationship between the stiffness matrix , nodal volumetric fluid flow rates and nodal potentials or fluid heads is given by,

$$[k]\{p\} = \{f\} ------(9)$$

Therefore the volumetric flow rate f in m^3/s is ,

$$f = v * A \quad ------(10)$$

Using equation (7), we obtain f in scalar form

$$f = -K_{xx} A g --------(11)$$

The hydraulic gradient matrix is,

$$g = \frac{p_2 - p_1}{L} ------(12)$$

Applying (11) and (12) at nodes 1 and 2, we obtain

$$f_1 = -k_{xx} A\left(\frac{p_2 - p_1}{L}\right) \text{ and}$$

$$f_2 = k_{xx} A\left(\frac{p_2 - p_1}{L}\right) ------(13)$$

Where f_1 is directed into the element ($p_1 > p_2$)

f_2 is directed away from the element indicating fluid is flowing out of the element.

In matrix form we write,

$$\begin{Bmatrix} f_1 \\ f_2 \end{Bmatrix} = \frac{Ak_{xx}}{L}\begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix}\begin{Bmatrix} p_1 \\ p_2 \end{Bmatrix} ---------(14)$$

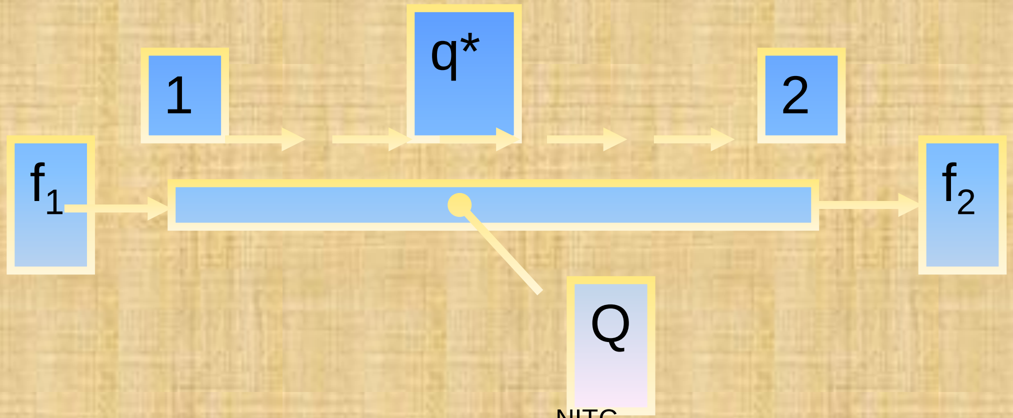
The element stiffness matrix is,

$$[\mathrm{k}] = \frac{\mathrm{A\,k_{xx}}}{\mathrm{L}}\begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix} ---------(15) \quad \mathrm{m^2/s}$$

Additional sources of volumetric fluid flow rates

In general, the basic element may be subjected to internal sources or sinks (such as from pump) or to surface-edge flow rates, such as from river or stream.

Consider the element with an internal source Q acting over the whole element & a uniform surface flow rate source q* acting over a surface as shown.



The force matrix terms are,

$$\{f_Q\} = \int_V [N]^T Q\, dV = \frac{QAL}{2}\begin{Bmatrix}1\\1\end{Bmatrix}\ m^3/s$$

Where Q will have units $m^3/(m^3\text{-}s)$ or 1/s

and

$$\{f_q\} = \int_{s_2} q^*[N]^T dS = \frac{q^* L t}{2}\begin{Bmatrix}1\\1\end{Bmatrix}\ m^3/s$$

q* will be in m /s.

Step 5. Assemble the element equations to obtain the global equations and introduce boundary conditions

$$[K] = \sum [k^e], \quad [F] = \sum [f^e] \; and \; \{F\} = [K]\{p\}$$

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For the assembly the requirement is that the potential at a common node between two elements be equal.

Step 6. Solve for the nodal potential.

Step 7. Solve for the element velocities and volumetric flow rates using

$$Q_f = v \, x \, A \quad m^3/s$$

Thank You