Fluid flow through porous media

Problems with solution

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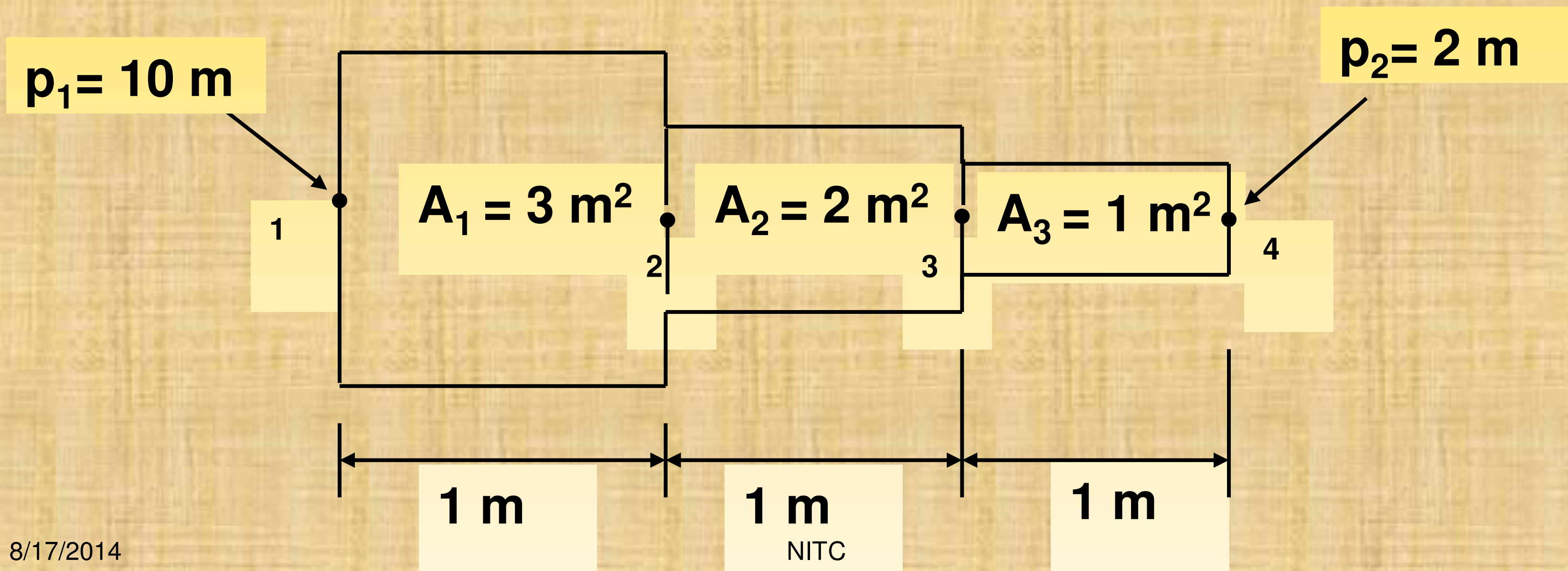
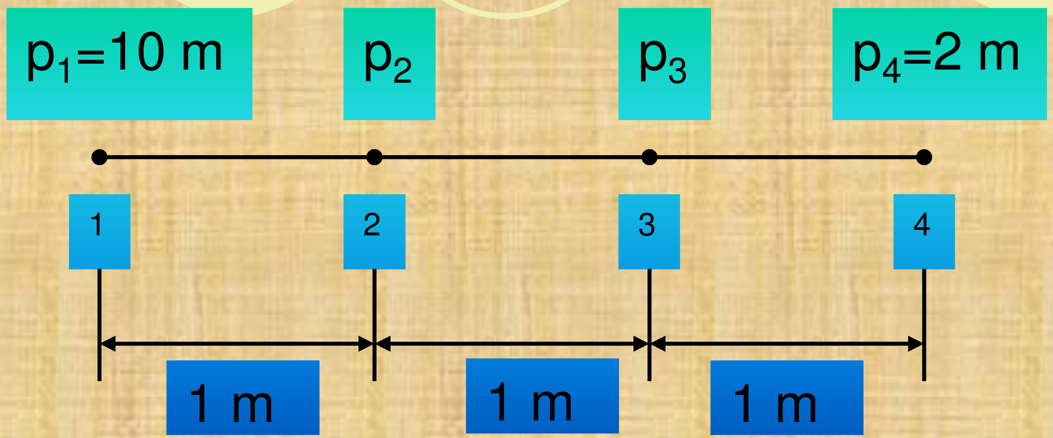
Problem 1.

For the smooth pipe of variable cross section shown in figure, determine the potentials at the junctions, the velocities in each pipe and volumetric flow rate. The potential at the left end is 10 m and that at the right end is 2 m.

The permeability co-efficient is 1 m / sec.



Finite element model:



The element stiffness matrices are,

$$[k^1] = \frac{A_1 K_1}{L_1}\begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix} = \frac{3x1}{1}\begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix} = \begin{array}{c} \begin{array}{cc} 1C & 2C \end{array} \\ \begin{bmatrix} 3 & -3 \\ -3 & 3 \end{bmatrix} \begin{array}{c} 1R \\ 2R \end{array} \end{array} \; m^2/s$$

$$[k^2] = \frac{A_2K_2}{L_2}\begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix} = \frac{2x1}{1}\begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix} = \left\| \begin{matrix} 2C & 3C & \\ \begin{bmatrix} 2 & -2 \\ -2 & 2 \end{bmatrix} & & \begin{matrix} 2R \\ 3R \end{matrix} \end{matrix} \right\| \ m^2/s$$

$$[k^3] = \frac{A_3K_3}{L_3}\begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix} = \frac{1x1}{1}\begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix} = \left\| \begin{matrix} 3C & 4C & \\ \begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix} & & \begin{matrix} 3R \\ 4R \end{matrix} \end{matrix} \right\| \ m^2/s$$

and the global stiffness matrix is,

$$[K] = \left\| \begin{matrix} 1C & 2C & 3C & 4C & \\ 3 & -3 & 0 & 0 & 1R \\ -3 & 3+2 & -2 & 0 & 2R \\ 0 & -2 & 2+1 & -1 & 3R \\ 0 & 0 & -1 & 1 & 4R \end{matrix} \right\| = \left\| \begin{matrix} 1C & 2C & 3C & 4C & \\ 3 & -3 & 0 & 0 & 1R \\ -3 & 5 & -2 & 0 & 2R \\ 0 & -2 & 3 & -1 & 3R \\ 0 & 0 & -1 & 1 & 4R \end{matrix} \right\|$$

In the given problem Q =0 and q* = 0, therefore,

$$\{f^1\} = \{f^2\} = \{f^3\} = 0$$

The global equations are,

$$\begin{bmatrix} 3 & -3 & 0 & 0 \\ -3 & 5 & -2 & 0 \\ 0 & -2 & 3 & -1 \\ 0 & 0 & -1 & 1 \end{bmatrix} \begin{Bmatrix} p_1 \\ p_2 \\ p_3 \\ p_4 \end{Bmatrix} = \begin{Bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{Bmatrix}$$

Nodal fluid head boundary conditions given are,

p_1 = 10 m , p_4 = 2 m

Modifying the stiffness (permeability) matrix and force matrix, we get

$$\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 5 & -2 & 0 \\ 0 & -2 & 3 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{Bmatrix} p_1 \\ p_2 \\ p_3 \\ p_4 \end{Bmatrix} = \begin{Bmatrix} 10 \\ 30 \\ 2 \\ 2 \end{Bmatrix}$$

We get two sets of simultaneous equations,

$$5p_2 - 2p_3 = 30$$

$$-2p_2 + 3p_3 = 2$$

On solving $p_2 = 8.546 \ and \ p_3 = 6.364$

Velocities in each element:

$$v_x^1 = -K_{xx}[B]\{p\} = -K_{xx}\begin{bmatrix} -\frac{1}{L} & \frac{1}{L} \end{bmatrix}\begin{Bmatrix} p_1 \\ p_2 \end{Bmatrix} = -1\begin{bmatrix} -\frac{1}{1} & \frac{1}{1} \end{bmatrix}\begin{Bmatrix} 10 \\ 8.546 \end{Bmatrix}$$

$$= -\begin{bmatrix} -1 & 1 \end{bmatrix}\begin{Bmatrix} 10 \\ 8.546 \end{Bmatrix} = 1.454 \text{ m/s}$$

$$v_x^2 = -K_{xx}\begin{bmatrix} -\frac{1}{L} & \frac{1}{L} \end{bmatrix}\begin{Bmatrix} p_2 \\ p_3 \end{Bmatrix} = -1\begin{bmatrix} -1 & 1 \end{bmatrix}\begin{Bmatrix} 8.546 \\ 6.364 \end{Bmatrix}$$

$$= 2.182 \text{ m/s}$$

$$v_x^3 = -K_{xx}\begin{bmatrix} -\frac{1}{L} & \frac{1}{L} \end{bmatrix}\begin{Bmatrix} p_3 \\ p_4 \end{Bmatrix} = -1\begin{bmatrix} -1 & 1 \end{bmatrix}\begin{Bmatrix} 6.364 \\ 2 \end{Bmatrix}$$

$$= 4.364 \text{ m/s}$$

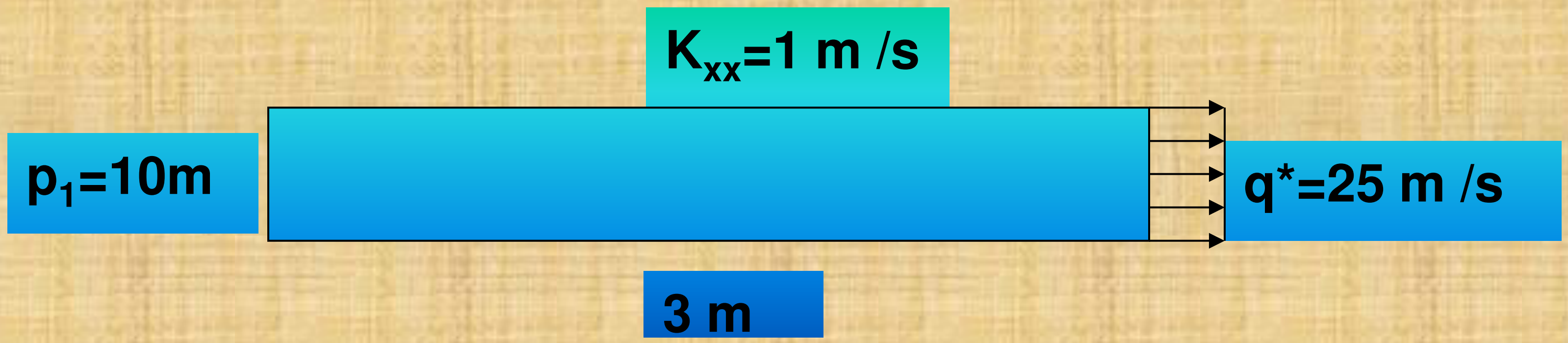
The volumetric flow rate is given by,

$$Q_f = A_1 \, x \, v_x^1 = 3 \, x \, 1.454 = 4.362 \ m^3/s$$

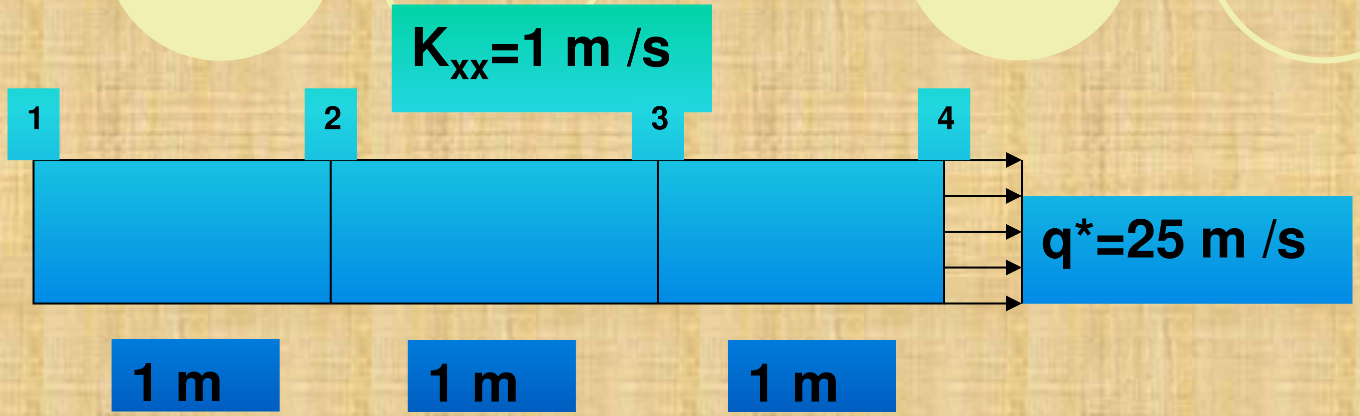
Problem 2.:

For the one-dimensional flow through the porous medium shown in figure with fluid flux at the right end, determine the potentials at the third points. Also determine the velocities in each element.

Let A = 2 m^2.



Finite Element model:



Element stiffness matrices are,

$$[k^1] = \frac{AK_{xx}}{L}\begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix} = \frac{2X1}{1}\begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix} = \left\| \begin{array}{cc|c} 1C & 2C & \\ 2 & -2 & 1R \\ -2 & 2 & 2R \end{array} \right\| \ \text{m/s}$$

$$[k^2] = \left\| \begin{matrix} & 2C & 3C & \\ & \begin{bmatrix} 2 & -2 \\ -2 & 2 \end{bmatrix} & & \begin{matrix} 2R \\ 3R \end{matrix} \end{matrix} \right\| \text{ m/s} \quad \text{and} \quad [k^3] = \left\| \begin{matrix} & 3C & 4C & \\ & \begin{bmatrix} 2 & -2 \\ -2 & 2 \end{bmatrix} & & \begin{matrix} 3R \\ 4R \end{matrix} \end{matrix} \right\| \text{ m/s}$$

And the global stiffness matrix is,

$$[K] = \left\| \begin{matrix} 1C & 2C & 3C & 4C & \\ 2 & -2 & 0 & 0 & 1R \\ -2 & 2+2 & -2 & 0 & 2R \\ 0 & -2 & 2+2 & -2 & 3R \\ 0 & 0 & -2 & 2 & 4R \end{matrix} \right\| = \left\| \begin{matrix} 1C & 2C & 3C & 4C & \\ 2 & -2 & 0 & 0 & 1R \\ -2 & 4 & -2 & 0 & 2R \\ 0 & -2 & 4 & -2 & 3R \\ 0 & 0 & -2 & 2 & 4R \end{matrix} \right\|$$

At node, q* = 25 m /s and $\{f_4\}$ will be,

$$\{f_4\} = A \times q^* = 2x25 = 50\,m^3/s$$ *(Negative)

The global equations are,

$$\left\|\begin{array}{c} \begin{array}{cccc} 1C & 2C & 3C & 4C \end{array} \\ \begin{bmatrix} 2 & -2 & 0 & 0 \\ -2 & 4 & -2 & 0 \\ 0 & -2 & 4 & -2 \\ 0 & 0 & -2 & 2 \end{bmatrix} \begin{array}{c} 1R \\ 2R \\ 3R \\ 4R \end{array} \end{array}\right\| \begin{Bmatrix} p_1 \\ p_2 \\ p_3 \\ p_4 \end{Bmatrix} = \begin{Bmatrix} f_1 \\ f_2 \\ f_3 \\ f_4 \end{Bmatrix}$$

Applying the known values of p and f, we get

$$
\begin{array}{c} \\ 1R \\ 2R \\ 3R \\ 4R \end{array}
\begin{array}{cccc} 1C & 2C & 3C & 4C \\ \left[\begin{array}{cccc} 1 & 0 & 0 & 0 \\ 0 & 4 & -2 & 0 \\ 0 & -2 & 4 & -2 \\ 0 & 0 & -2 & 2 \end{array}\right] \end{array}
\begin{Bmatrix} p_1 \\ p_2 \\ p_3 \\ p_4 \end{Bmatrix} = \begin{Bmatrix} 10 \\ 20 \\ 0 \\ -50 \end{Bmatrix}
$$

The equations are,

$$p_1 = 10$$

$$4p_2 - 2p_3 = 20 \text{ -----------(1)}$$

$$-2p_2 + 4p_3 - 2p_4 = 0 \text{ -----------(2)}$$

$$-2p_3 + 2p_4 = -50 \text{ -----------(3)}$$

On solving

$$p_1 = 10\,m\,(given)\;,\;\; p_2 = -15\,m\;,\; p_3 = -40\,m,\;\; p_4 = -65\,m$$

Therefore, $\{p\}^T = [10 \quad -15 \quad -40 \quad -65]\,m$

Velocities in each element:

$$v_x^1 = -K_{xx}[B]\{p\} = -K_{xx}\begin{bmatrix} -\frac{1}{L} & \frac{1}{L} \end{bmatrix}\begin{Bmatrix} p_1 \\ p_2 \end{Bmatrix} = -1\begin{bmatrix} -\frac{1}{1} & \frac{1}{1} \end{bmatrix}\begin{Bmatrix} 10 \\ -15 \end{Bmatrix}$$

$$= -\begin{bmatrix} -1 & 1 \end{bmatrix}\begin{Bmatrix} 10 \\ -15 \end{Bmatrix} = 25\,m/s$$

$$v_x^2 = -K_{xx}\begin{bmatrix} -\frac{1}{L} & \frac{1}{L} \end{bmatrix}\begin{Bmatrix} p_2 \\ p_3 \end{Bmatrix} = -1\begin{bmatrix} -\frac{1}{1} & \frac{1}{1} \end{bmatrix}\begin{Bmatrix} -15 \\ -40 \end{Bmatrix}$$

$$= -\begin{bmatrix} -1 & 1 \end{bmatrix}\begin{Bmatrix} -15 \\ -40 \end{Bmatrix} = 25\,\text{m/s}$$

$$v_x^3 = -K_{xx}\begin{bmatrix} -\frac{1}{L} & \frac{1}{L} \end{bmatrix}\begin{Bmatrix} p_3 \\ p_4 \end{Bmatrix} = -1\begin{bmatrix} -\frac{1}{1} & \frac{1}{1} \end{bmatrix}\begin{Bmatrix} -40 \\ -65 \end{Bmatrix}$$

$$= -\begin{bmatrix} -1 & 1 \end{bmatrix}\begin{Bmatrix} -40 \\ -65 \end{Bmatrix} = 25\,\text{m/s}$$

$$\text{i.e. } v_x^1 = v_x^2 = v_x^3 = 25 \text{ m/s}$$

The volume flow rate is given by,

$$Q_f = A\, v_x^1 = 2 \times 25 = 50 \,\text{m}^3/\text{s}$$

Thank You